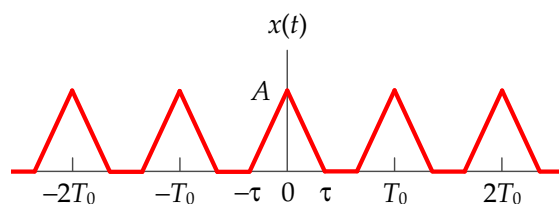
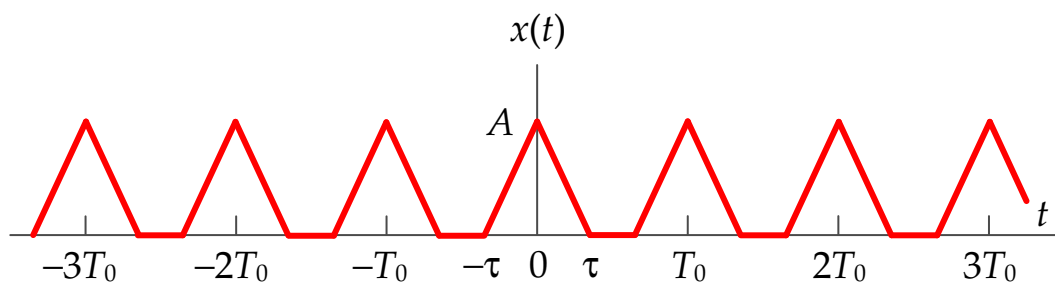


15 Fourier Series: More Examples

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Q1. For the following signal $x(t) = \text{rep}_{T_0} \left\{ A \Delta \left(\frac{t}{\tau} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q1. Solution. The complex exponential Fourier series coefficients

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\begin{aligned} \alpha_n &= \frac{1}{T_0} \int_{-T_0/2}^{-\tau} (0) dt + \frac{1}{T_0} \int_{-\tau}^{\tau} \left(A - \left| \frac{A}{\tau} t \right| \right) (e^{-jn\omega_0 t}) dt + \frac{1}{T_0} \int_{\tau}^{T_0/2} (0) dt \\ &= \frac{1}{T_0} \int_{-\tau}^0 \left(A + \frac{A}{\tau} t \right) (e^{-jn\omega_0 t}) dt + \frac{1}{T_0} \int_0^{\tau} \left(A - \frac{A}{\tau} t \right) (e^{-jn\omega_0 t}) dt \end{aligned}$$

$$\begin{aligned} \alpha_n &= \frac{A}{T_0} \int_{-\tau}^0 e^{-jn\omega_0 t} dt + \frac{A}{\tau T_0} \int_{-\tau}^0 (t) (e^{-jn\omega_0 t}) dt \\ &\quad + \frac{A}{T_0} \int_0^{\tau} e^{-jn\omega_0 t} dt + \frac{-A}{\tau T_0} \int_0^{\tau} (t) (e^{-jn\omega_0 t}) dt \end{aligned}$$

$$\begin{aligned} \alpha_0 &= \frac{A}{T_0} \int_{-\tau}^0 (e^0) dt + \frac{A}{\tau T_0} \int_{-\tau}^0 (t) (e^0) dt + \frac{A}{T_0} \int_0^{\tau} (e^0) dt + \frac{-A}{\tau T_0} \int_0^{\tau} (t) (e^0) dt \\ &= \frac{A}{T_0} [t]_{-\tau}^0 + \frac{A}{\tau T_0} \left[\frac{t^2}{2} \right]_{-\tau}^0 + \frac{A}{T_0} [t]_0^{\tau} + \frac{-A}{\tau T_0} \left[\frac{t^2}{2} \right]_0^{\tau} \\ &= \frac{A}{T_0} [0 - (-\tau)] + \frac{A}{\tau T_0} \left[\frac{0^2}{2} - \frac{(-\tau)^2}{2} \right] + \frac{A}{T_0} [(\tau) - 0] + \frac{-A}{\tau T_0} \left[\frac{(\tau)^2}{2} - \frac{0^2}{2} \right] \end{aligned}$$

$$\alpha_0 = \frac{A\tau}{T_0} + \frac{-A\tau}{2T_0} + \frac{A\tau}{T_0} + \frac{-A\tau}{2T_0} = \frac{2A\tau}{T_0} - \frac{A\tau}{T_0}$$

$$\alpha_0 = \frac{A\tau}{T_0}$$

$$\begin{aligned}
\alpha_n &= \frac{A}{T_0} \int_{-\tau}^0 (e^{-jn\omega_0 t}) dt + \frac{A}{\tau T_0} \int_{-\tau}^0 (t)(e^{-jn\omega_0 t}) dt \\
&\quad + \frac{A}{T_0} \int_0^{\tau} (e^{-jn\omega_0 t}) dt + \frac{-A}{\tau T_0} \int_0^{\tau} (t)(e^{-jn\omega_0 t}) dt \\
&= \frac{A}{T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_{-\tau}^0 + \frac{A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [(-jn\omega_0 t - 1)e^{-jn\omega_0 t}]_{-\tau}^0 \\
&\quad + \frac{A}{T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_0^{\tau} + \frac{-A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [(-jn\omega_0 t - 1)e^{-jn\omega_0 t}]_0^{\tau}
\end{aligned}$$

Note: Integration by parts $\int u dv = uv - \int v du$

$$\int (t)(e^{at}) dt = e^{at} \left(\frac{at - 1}{a^2} \right)$$

$$\begin{aligned}
\alpha_n &= \frac{A}{T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_{-\tau}^0 + \frac{A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [(-jn\omega_0 t - 1)e^{-jn\omega_0 t}]_{-\tau}^0 \\
&\quad + \frac{A}{T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_0^{\tau} + \frac{-A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [(-jn\omega_0 t - 1)e^{-jn\omega_0 t}]_0^{\tau} \\
\alpha_n &= \frac{A}{-jn\omega_0 T_0} [1 - e^{jn\omega_0 \tau}] + \frac{A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [-1 - (jn\omega_0 \tau - 1)e^{jn\omega_0 \tau}] \\
&\quad + \frac{A}{-jn\omega_0 T_0} [e^{-jn\omega_0 \tau} - 1] + \frac{-A}{\tau T_0} \times \frac{1}{(jn\omega_0)^2} [(-jn\omega_0 \tau - 1)e^{-jn\omega_0 \tau} + 1] \\
\alpha_n &= \frac{A}{-jn\omega_0 T_0} [1 - 1] + \frac{A}{-jn\omega_0 T_0} [e^{-jn\omega_0 \tau} - e^{jn\omega_0 \tau}] \\
&\quad + \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + (1 - jn\omega_0 \tau)e^{jn\omega_0 \tau} + (1 + jn\omega_0 \tau)e^{-jn\omega_0 \tau}]
\end{aligned}$$

$$\alpha_n = \frac{A}{-jn\omega_0 T_0} [1 - 1] + \frac{A}{-jn\omega_0 T_0} [e^{-jn\omega_0\tau} - e^{jn\omega_0\tau}]$$

$$+ \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + (1 - jn\omega_0\tau)e^{jn\omega_0\tau} + (1 + jn\omega_0\tau)e^{-jn\omega_0\tau}]$$

$$\alpha_n = \frac{A}{jn\omega_0 T_0} [e^{jn\omega_0\tau} - e^{-jn\omega_0\tau}]$$

$$+ \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + (1 - jn\omega_0\tau)e^{jn\omega_0\tau} + (1 + jn\omega_0\tau)e^{-jn\omega_0\tau}]$$

$$\alpha_n = \frac{A}{(jn\omega_0)^2 T_0} [jn\omega_0 e^{jn\omega_0\tau} - jn\omega_0 e^{-jn\omega_0\tau}]$$

$$+ \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + e^{jn\omega_0\tau} - jn\omega_0\tau e^{jn\omega_0\tau} + e^{-jn\omega_0\tau} + jn\omega_0\tau e^{-jn\omega_0\tau}]$$

$$\alpha_n = \frac{A}{(jn\omega_0)^2 T_0} [jn\omega_0 e^{jn\omega_0\tau} - jn\omega_0 e^{-jn\omega_0\tau}]$$

$$+ \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + e^{jn\omega_0\tau} - jn\omega_0\tau e^{jn\omega_0\tau} + e^{-jn\omega_0\tau} + jn\omega_0\tau e^{-jn\omega_0\tau}]$$

$$\alpha_n = \frac{A}{\tau T_0 (jn\omega_0)^2} [-1 - 1 + e^{jn\omega_0\tau} + e^{-jn\omega_0\tau}]$$

$$= \frac{A e^{-jn\omega_0\tau}}{\tau T_0 (jn\omega_0)^2} [-e^{jn\omega_0\tau} - e^{jn\omega_0\tau} + e^{j2n\omega_0\tau} + 1]$$

$$= \frac{A e^{-jn\omega_0\tau}}{\tau T_0 (jn\omega_0)^2} [e^{jn\omega_0\tau} - 1]^2$$

Because $(a - 1)^2 = (a - 1)(a - 1) = a^2 - 2a + 1$

$$\begin{aligned}
\alpha_n &= \frac{A e^{-jn\omega_0\tau}}{\tau T_0 (jn\omega_0)^2} [e^{jn\omega_0\tau} - 1]^2 \\
\alpha_n &= \frac{A e^{-jn\omega_0\tau}}{\tau T_0 (jn\omega_0)^2} [e^{jn\omega_0\tau/2} (e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2})]^2 \\
&= \frac{A e^{-jn\omega_0\tau} e^{jn\omega_0\tau}}{\tau T_0 (jn\omega_0)^2} [e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}]^2 \\
&= \frac{A}{\tau T_0 (jn\omega_0)^2} \left[2j \sin\left(\frac{n\omega_0\tau}{2}\right) \right]^2 \\
\alpha_n &= \frac{A \tau^2}{\tau T_0} \left[\frac{\sin\left(\pi \frac{n\omega_0\tau}{2\pi}\right)}{\pi \frac{n\omega_0\tau}{2\pi}} \right]^2 = \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right), \quad n \neq 0
\end{aligned}$$

So,

$$\alpha_n = \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) = \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\tau}{T_0}\right), \quad n = 0, \pm 1, \pm 2, \dots$$

Notice that $\alpha_0 = \frac{A\tau}{T_0} \text{sinc}^2(0) = \frac{A\tau}{T_0}$. For this signal, the values of α_n are real and always positive, and hence, the phase of α_n is 0° .

$$\begin{aligned}
|\alpha_n| &= \left| \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) \right| = \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) \\
\angle \alpha_n &= 0^\circ
\end{aligned}$$

Since $x(t)$ is real-valued signal, we have

$$\alpha_{-n} = \alpha_n^* , \quad n = 0, 1, 2, 3, \dots$$

$$|\alpha_{-n}| = \left| \frac{A\tau}{T_0} \operatorname{sinc}^2 \left(\frac{n\omega_0\tau}{2\pi} \right) \right| = \frac{A\tau}{T_0} \operatorname{sinc}^2 \left(\frac{n\omega_0\tau}{2\pi} \right)$$

$$\angle \alpha_{-n} = 0^\circ$$

The average value or DC value is

$$\alpha_0 = \frac{A\tau}{T_0} \operatorname{sinc}^2(0) = \frac{A\tau}{T_0}$$

The trigonometric Fourier series is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] , \quad \omega_0 = \frac{2\pi}{T_0}$$

where the coefficients are calculated as

$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

Since $x(t)$ is even, then $b_n = 0, \forall n$, and $a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t) dt$

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = \frac{A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) - j0, \quad n = 0, 1, 2, 3, \dots$$

$$a_n = 2 \text{Re}\{\alpha_n\}, \quad n = 0, 1, 2, 3, \dots$$

$$b_n = -2 \text{Im}\{\alpha_n\}, \quad n = 0, 1, 2, 3, \dots$$

Hence,

$$a_0 = \frac{2A\tau}{T_0}$$

$$a_n = \frac{2A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right)$$

$$b_n = 0$$

The compact Fourier series is

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \varphi_n), \quad \omega_0 = \frac{2\pi}{T_0}$$

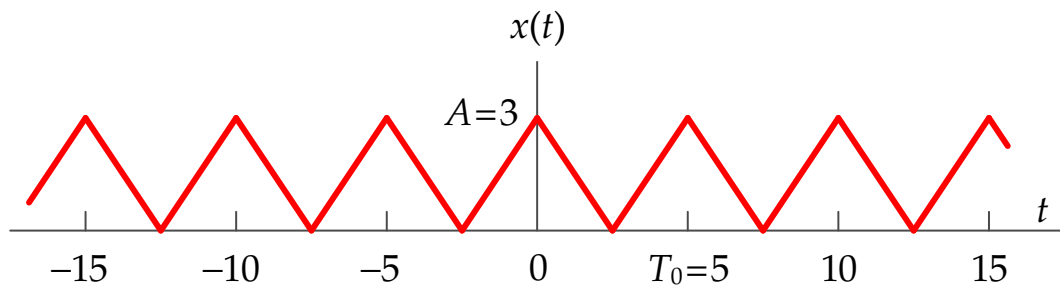
$$c_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2 + 0^2} = |a_n| = \left| \frac{2A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) \right|$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{0}{a_n}\right) = 0^\circ$$

$$c_n = 2|\alpha_n|$$

$$\theta_n = -\angle\alpha_n = \tan^{-1}\left(\frac{-\text{Im}\{\alpha_n\}}{\text{Re}\{\alpha_n\}}\right), \quad \varphi_n = \angle\alpha_n$$

Q2. For the following signal $x(t) = \text{rep}_5 \left\{ 3 \Delta \left(\frac{t}{2.5} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q2. Solution. Please do the integrals again, or just substitute to get

$$\alpha_n = \frac{A\tau}{T_0} \text{sinc}^2 \left(\frac{n\omega_0\tau}{2\pi} \right) = \frac{A\tau}{T_0} \text{sinc}^2 \left(\frac{n\tau}{T_0} \right) = \frac{3 \times 2.5}{5} \text{sinc}^2 \left(\frac{n \times 2.5}{5} \right) = \frac{3}{2} \text{sinc}^2 \left(\frac{n}{2} \right)$$

$$a_n = \frac{2A\tau}{T_0} \text{sinc}^2 \left(\frac{n\omega_0\tau}{2\pi} \right) = 3 \text{sinc}^2 \left(\frac{n}{2} \right)$$

$$b_n = 0$$

The average value is $\alpha_0 = \frac{3}{2} \text{sinc}^2 \left(\frac{0}{2} \right) = \frac{3}{2} = 1.5$.

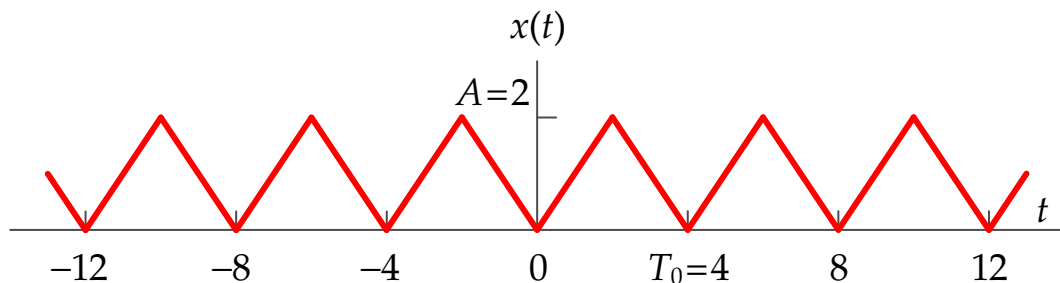
$$c_n = 2|\alpha_n|$$

$$\theta_n = -\angle\alpha_n = \tan^{-1} \left(\frac{-\text{Im}\{\alpha_n\}}{\text{Re}\{\alpha_n\}} \right)$$

$\alpha_0 = \frac{3}{2} \text{sinc}^2\left(\frac{0}{2}\right) = \frac{3}{2}$	$ \alpha_0 = 1.5$	$\angle\alpha_0 = 0$
$\alpha_1 = \frac{3}{2} \text{sinc}^2\left(\frac{1}{2}\right) = \frac{3 \sin^2(\pi/2)}{2 (\pi/2)^2} = \frac{6}{\pi^2}$	$ \alpha_1 = 0.6079$	$\angle\alpha_1 = 0$
$\alpha_2 = \frac{3}{2} \text{sinc}^2\left(\frac{2}{2}\right) = \frac{3 \sin^2(\pi)}{2 (\pi)^2} = \frac{3 \times 0}{2\pi^2}$	$ \alpha_2 = 0$	$\angle\alpha_2 = 0$
$\alpha_3 = \frac{3}{2} \text{sinc}^2\left(\frac{3}{2}\right) = \frac{3 \sin^2(3\pi/2)}{2 (3\pi/2)^2} = \frac{2}{3\pi^2}$	$ \alpha_3 = 0.0675$	$\angle\alpha_3 = 0$
$\alpha_4 = \frac{3}{2} \text{sinc}^2\left(\frac{4}{2}\right) = \frac{3 \sin^2(2\pi)}{2 (2\pi)^2} = \frac{3 \times 0}{8\pi^2}$	$ \alpha_4 = 0$	$\angle\alpha_4 = 0$
$\alpha_5 = \frac{3}{2} \text{sinc}^2\left(\frac{5}{2}\right) = \frac{3 \sin^2(5\pi/2)}{2 (5\pi/2)^2} = \frac{6}{25\pi^2}$	$ \alpha_5 = 0.0243$	$\angle\alpha_5 = 0$

....

Q3. For the following signal $x(t) = \text{rep}_4 \left\{ 2 \Delta \left(\frac{t-2}{2} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q3. Solution. Notice that $x(t)$ is an even signal. Hence,

$$a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t) dt = \frac{4}{T_0} \int_0^{T_0/2} \left(\frac{2A}{T_0} t\right) \cos(n\omega_0 t) dt$$

$$\begin{aligned} a_0 &= \frac{4}{T_0} \int_0^{T_0/2} \left(\frac{2A}{T_0} t\right) \cos(0) dt = \frac{4}{T_0} \times \frac{2A}{T_0} \int_0^{T_0/2} t dt = \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{t^2}{2}\right]_0^{T_0/2} \\ &= \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{(T_0/2)^2}{2} - \frac{0^2}{2}\right] = A \end{aligned}$$

Hence, the average value (DC value) is

$$\alpha_0 = \frac{a_0}{2} = \frac{A}{2}$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} \left(\frac{2A}{T_0} t\right) \cos(n\omega_0 t) dt = \frac{4}{T_0} \times \frac{2A}{T_0} \int_0^{T_0/2} t \cos(n\omega_0 t) dt$$

Note: Integration by parts $\int u dv = uv - \int v du$

$$\int t \cos(at) dt = \frac{1}{a^2} \cos(at) + \frac{t}{a} \sin(at)$$

$$a_n = \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{1}{(n\omega_0)^2} \cos(n\omega_0 t) + \frac{t}{n\omega_0} \sin(n\omega_0 t) \right]_0^{T_0/2}$$

$$\begin{aligned}
a_n &= \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{1}{(n\omega_0)^2} \cos(n\omega_0 t) + \frac{t}{n\omega_0} \sin(n\omega_0 t) \right]_0^{T_0/2} \\
&= \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{1}{(n\omega_0)^2} \cos\left(\frac{n\omega_0 T_0}{2}\right) + \frac{T_0/2}{n\omega_0} \sin\left(\frac{n\omega_0 T_0}{2}\right) - \frac{1}{(n\omega_0)^2} \right] \\
&= \frac{4}{T_0} \times \frac{2A}{T_0} \left[\frac{1}{(n\omega_0)^2} \cos(n\pi) + \frac{T_0/2}{n\omega_0} \sin(n\pi) - \frac{1}{(n\omega_0)^2} \right] \\
&= \frac{4}{T_0} \times \frac{2A}{T_0} \times \frac{1}{(n\omega_0)^2} [\cos(n\pi) - 1]
\end{aligned}$$

$$a_n = \frac{2A}{(n\pi)^2} [\cos(n\pi) - 1], \quad n \neq 0$$

$$b_n = 0, \quad \forall n$$

Hence,

$$a_n = \frac{2A}{(n\pi)^2} [\cos(n\pi) - 1] = 0, \quad \forall n = \text{even}$$

$$a_n = \frac{2A}{(n\pi)^2} [\cos(n\pi) - 1] = \frac{-4A}{(n\pi)^2}, \quad \forall n = \text{odd}$$

$$b_n = 0, \quad \forall n$$

Also,

$$\alpha_0 = \frac{a_0}{2} = \frac{A}{2}$$

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = \frac{A}{(n\pi)^2} [\cos(n\pi) - 1] - j0$$

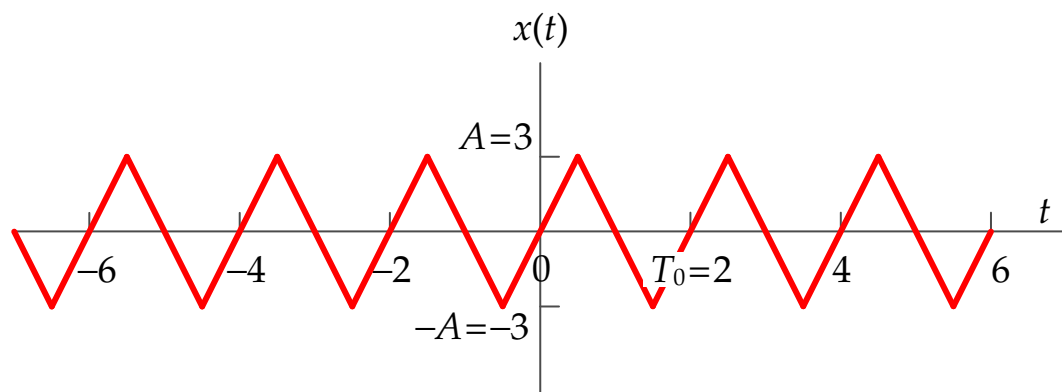
Alternatively, we can use Fourier transform (and Fourier series) properties to quickly come up with the second equation,

$$\begin{aligned}\alpha_n &= \frac{A\tau}{T_0} \operatorname{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) e^{-jn\omega_0 t_0} = \frac{AT_0/2}{T_0} \operatorname{sinc}^2\left(\frac{n\omega_0 T_0/2}{2\pi}\right) e^{-jn\omega_0 T_0/2} \\ &= \frac{A}{2} \operatorname{sinc}^2\left(\frac{n}{2}\right) e^{-jn\pi}, \quad n \neq 0\end{aligned}$$

The compact Fourier series is

$$\begin{aligned}c_n &= 2|\alpha_n| = \sqrt{a_n^2 + b_n^2} = |a_n| \\ \theta_n &= -\angle\alpha_n = \tan^{-1}\left(\frac{-\operatorname{Im}\{\alpha_n\}}{\operatorname{Re}\{\alpha_n\}}\right) = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{0}{a_n}\right) = n\pi \equiv \pi\end{aligned}$$

Q4. For the following signal $x(t) = \operatorname{rep}_2\{6 \Delta(t - 0.5)\} - 3$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q4. Solution.

$$a_0 = 0$$

$$a_n = 0, \quad n \neq 0$$

$$b_n = \frac{8A}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even} \\ \frac{8A}{(n\pi)^2}, & n = 1, 5, 9, 13, \dots \\ \frac{-8A}{(n\pi)^2}, & n = 3, 7, 11, 15, \dots \end{cases}$$

$$\alpha_0 = \frac{a_0}{2} = 0$$

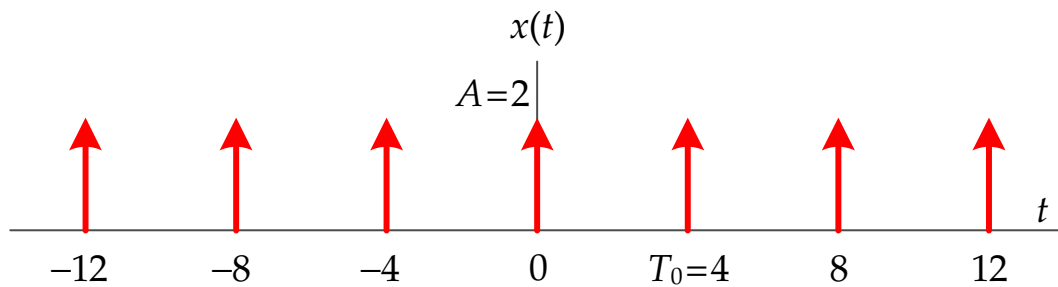
$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = 0 - j \frac{4A}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right), \quad n \neq 0$$

Alternatively, from Fourier transform (and Fourier series) properties,

$$\begin{aligned} \alpha_n &= \frac{2A\tau}{T_0} \text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) e^{-jn\omega_0 t_0} = \frac{2AT_0/2}{T_0} \text{sinc}^2\left(\frac{n\omega_0 T_0/2}{2\pi}\right) e^{-jn\omega_0 T_0/4} \\ &= A \text{sinc}^2\left(\frac{n}{2}\right) e^{-j\frac{n\pi}{2}}, \quad n \neq 0 \end{aligned}$$

$$|\alpha_n| = \frac{c_n}{2} = \begin{cases} 0, & n \text{ even} \\ \frac{4A}{(n\pi)^2}, & n \text{ odd} \end{cases} \quad \angle\alpha_n = -\theta_n = \frac{-n\pi}{2} = \begin{cases} 0, & n \text{ even} \\ -\pi/2, & n = 1, 5, 9, 13, \dots \\ \pi/2, & n = 3, 7, 11, 15, \dots \end{cases}$$

Q5. For the following signal $x(t) = \text{rep}_{T_0}\{A \delta(t)\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q1. Solution. The complex exponential Fourier series coefficients

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A\delta(t) e^{-jn\omega_0 t} dt = \frac{A}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt$$

An important property of the impulse function (called the sampling property) is

$$\int_{-\infty}^{\infty} x(t) \delta(t - t_0) dt = x(t_0) \int_{-\infty}^{\infty} \delta(t - t_0) dt = x(t_0)$$

$$\alpha_n = \frac{A}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{A}{T_0} e^{-jn\omega_0 \times 0} = \frac{A}{T_0}, \quad n = 0, \pm 1, \pm 2, \dots$$

$$|\alpha_n| = \left| \frac{A}{T_0} \right| = \frac{A}{T_0}$$

$$\angle \alpha_n = 0^\circ$$

Since $x(t)$ is real-valued signal, we have

$$\alpha_{-n} = \alpha_n^*, \quad n = 0, 1, 2, 3, \dots$$

$$|\alpha_{-n}| = \left| \frac{A}{T_0} \right| = \frac{A}{T_0}$$

$$\angle \alpha_{-n} = 0^\circ$$

The trigonometric Fourier series is

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = \frac{A}{T_0} - j0, \quad n = 0, 1, 2, 3, \dots$$

Hence,

$$a_0 = \frac{2A}{T_0}$$

$$a_n = \frac{2A}{T_0}$$

$$b_n = 0$$